Complete Prime Modules for Graded Simple Modules Which is Simple Over Leavitt Path Algebra

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Abstract
We characterize a graded simple module $A_u$ in the setting Leavitt path algebras $L$ which is the completely prime module. Let $m \in M$ and $r \in A$, an $A$-module $M$ is called a completely prime module if $rm = 0$ for one $m \in M$ and $r \in A$ then $rM = 0$ or $m = 0$. In this article, we show that If $u$ is a sink or an infinite emitter, then a graded simple module which is simple $A_u$ is not a completely prime module.

INTRODUCTION

The Leavitt path algebra is a branch of representation theory. The Leavitt path algebra was introduced by Abrams and Aranda Pino in 2005 (Abrams & Aranda Pino, 2005), and has become an active area of research recently. Leavitt path algebra is a path algebra on an extended graph that satisfies a certain relation. Let $Q$ is a Quiver, the path algebra $KQ$ is an algebra over a field $K$ with basis is the set of all paths in $Q$. A quiver $Q = (Q^0, Q^1, r, s)$ are quadruple of the sets $Q^0$ which elements are called vertices, the set $Q^1$ which elements are called arrows, and two maps $r, s : Q^1 \to Q^0$, which associate to each arrow $e \in Q^1$, its source $s(e) \in Q^0$, and its range $r(e) \in Q^0$, respectively.

Many studies have been carried out on Leavitt path algebra (Abrams et al., 2015; Abrams & Aranda Pino, 2005; Ara & Pardo, 2017; Ara & Rangaswamy, 2014; Arnone & Cortiñas, 2022; Clark et al., 2016; Hay et al., 2014; Rangaswamy, 2015). Abrams et al in 2007 have studied the necessary and sufficient conditions for a quiver $Q$ so that the Leavitt path algebra is a finite-dimensional algebra. Ranggaswamy (Rangaswamy, 2015) studied the prime ideal characterization of the Leavitt path algebra of any graph. Ara and Rangaswamy (Ara & Rangaswamy, 2014) have also proved that a simple module on Leavitt path algebra is a finitely presented module. (Abrams & Aranda Pino, 2005)

The prime module concept was constructed from the structure of prime ideal of a ring [4]. Let $M$ be a left module over the ring $R$ (written $R$ - module ). A proper prime submodule $N$ of $M$ if $rM = 0$ with $r \in R$, and $m \in M$ implies $m \in N$ or $rM \subseteq N$ (Risnawita et al., 2021). Wardhana (Wardhana et al., 2021) and Saleh (Saleh et al., 2016) were characterized as prime submodules. A module $M$ (Risnawita et al., 2021) is said to be a completely prime module over ring $R$ if $rm = 0$ with $r \in R$ and $m \in M$ implies that $rM = 0$. The purpose of this study was to
characterize graded simple modules over Leavitt path algebra.

In this paper we will characterize graded simple modules which is simple for a Leavitt path algebras by using a quiver with a sink or infinite emitter. Let $Q$ be a quiver, A vertex $v$ in quiver $Q$ that does not emit an arrow, we call a sink. A vertex $v$ that shoots infinitely many arrows is called an infinite emitter.

In Section 2 of this paper, we will give the method that we use in this research. In section 3, we will give the result and discussion. In Section 4, we give a conclusion about graded simple modules which is simple for a Leavitt path algebras.

**RESULT AND DISCUSSION**

A quiver $Q = (Q^0, Q^1, r, s)$ consists of two sets $Q^0, Q^1$ and two maps $s: Q^1 → Q^0$ and $r: Q^1 → Q^0$. The study The Quiver was studied in papers (Abrams et al., 2011; Pino et al., 2010). If $Q^0$ is finite and $Q^1$ is finite then $Q$ is a finite quiver. If $s^{-1}(v)$ is finite for every $v ∈ Q^0$, then $Q$ is said to be a row finite quiver. A vertex $v$ that does not emit an arrow, we call a sink. A vertex $v$ that shoots infinitely many arrows is called an infinite emitter.

In quiver $Q$, a path is ordered arrows $p = e_1 e_2 ... e_n$, with $r(e_i) = s(e_{i+1})$ for all $i$. A finite path in quiver $Q$ is an ordered arrows $p = e_1 e_2 ... e_n$ where $r(e_i) = s(e_{i+1})$, for all $i$. In this case, the length of path $p$ is $n$, denoted by $l(p) = n$. Two infinite paths $p$ and $q$ are tail equivalent, denoted by $p ∼ q$ if $τ ≥_n (p) = τ ≥_m (q)$, for some integers $m,n$.

A path algebra $A = KQ$ is well-known in representation theory (Pino et al., 2010). A finite quiver $Q$ can be represented as a module. A representation $M$ of $Q$ can be defined: each vertex $a ∈ Q^0$ was associated with a $K$-vector space $M_a$ and each arrow $a: a → b$ in $Q^1$ associated to a $K$-linear map $φ_a : M_a → M_b$.

Because of a quiver, $Q$ can be seen as a directed graph, so we can add the edges in opposite direction. The arrows in $Q^1$ denoted as the real edge, and then the edges on the opposite are called the ghost edges. The set of all ghost edges in $Q$ is denoted as $(Q^1)^*$. Following (Chen, 2015) we give a definition of leavitt path algebra $L$ of quiver $Q$, let $k$ be coefficients as the $k$-algebra generated by a set $\{v : v ∈ Q^0\}$ and $\{e, e^* : e ∈ Q^1\}$ which satisfies the following relations:

1. $s(e)e = e = e r(e)$, for all $e ∈ Q^1$
2. $r(e)e^* = e^* e = e^* s(e)$, for all $e^* ∈ Q^1$
3. $(CK1) e^* f = δ_{ef}(e)$, for all $e, f ∈ Q^1$
4. $(CK2) v = ∑_{(e ∈ Q^1) | s(e) = v} e e^*$, whenever $v ∈ E^0$ is not a sink.

We denoted this algebra by $L_K(Q)$. Moreover, the relation (3) and (4) are called the Cuntz Krieger relations. For further, the Leavitt path algebra $L_K(Q)$ will be written by $L$. Every element of $L$ can be written as $a = ∑_{i=0}^{n} k_i α_i β_i^*$, where $k_i ≠ 0$, $k_i ∈ K$, and $α_i β_i$ are paths in $Q$.

Let $M$ is a left $L$ – module, for each $m ∈ M$, we will define the $L – homomorphis ρ : L → M$, with $ρ_m (r) = rm$. Through $ρ$, Chen has introduced some classes of simple modules over Leavitt path algebras (Chen, 2015). By using the Chens method, we will construct simple modules over $L$ by using sinks, infinite emitter, or cycles in the quiver $Q$ (Ara & Rangaswamy, 2014; Rangaswamy, 2015, 2020).

**Definition 3.1.** Let $u$ is a sink or an infinite emitter in a quiver $Q$. Let $A_u$ be the $K$ – vector space having as basis the set $B = \{p : p paths in Q with r (p) = u\}$. We construct $A$ a left $L$ – module as follows: for each vertex $v$, we define linear transformations $P_v$ and each edge $e$ in $Q$, linear transformations $S_e$, and $S_e^*$.
on $A$ by defining their actions on basis $B$ as follows:

1. For all $p \in B$,
2. $p_v(p) = \{ p, \text{if } v = s(p) \}
3. \begin{cases} ep, \text{if } r(e) = s(p) \end{cases}$
4. $0, \text{others}$
5. $0, \text{others}$

So it can be easily checked that the endomorphisms $\{ p_v, s_e, S_e^\ast : u \in Q^\ast, e \in Q^1 \}$ fulfill relations (1) - (4) of the Leavitt path algebra $L$. This induces an algebra homomorphism $\varphi$ from $L$ to $\text{End}_K(S_{v_0})$, mapping $u$ to $p_u$, $e$ to $S_e$ and $e^\ast$ to $S_e^\ast$. Then we can construct $A_u$ as a left module over $L$ through the homomorphism $\varphi$.

Furthermore, the following lemma explains that if $u$ is an infinite emitter or a sink then $A_u$ is a simple module.

**Lemma 3.2.** Let $u$ either a sink or an infinite emitter, if $u$ is those vertices then $A_u$ is a simple left $L$ module.

In (Hazrat, 2013) simple modules have been constructed graded simple modules over $L$ by using the $Q$ -grazed algebraic branching system. The graded simple modules that have been constructed is a graded simple module that is also simple.

**Definition 3.3.** Let $Q$ be an arbitrary quiver. The $Q$ -algebraic branching system consists of a set $X$ and a family of its subsets $\{ X_v, X_w, v \in Q^0, e \in Q^1 \}$ such that

1. $X_v \cap X_w = \emptyset = X_v \cap X_f$, for $v, w \in Q^0$, with $v \neq w$ and $e, f \in Q^1$ with $e \neq f$
2. $X_v \subseteq X_{s(e)}$ for $e \in Q^1$
3. For all $v \in Q^0$, $X_v = \bigcup_{e \in s^{-1}(v)} X_e$, for all $e \in s^{-1}(v)$ $X_e$;
4. For each $e \in Q^1$, there exists a bijection $\sigma_e : X_{r(e)} \rightarrow X_e$.

We assume that $X$ is saturated, that is, $X = \bigcup_{v \in Q^0} X_v$

Next, given the definition of algebraic branching system.

**Definition 3.4.** If there is a map $\text{deg} : X \rightarrow Z$ then The $Q$-branching system is a graded such that

$$ \text{deg}(\sigma_e(x)) = \text{deg}(x) + 1 \text{ for all } e.$$ Let $Q$ -algebraic branching system $X$, we can make a left module $M(X)$ over $L$ as follows:

Let $K$-vector space have $X$ as a basis. Following (Hazrat, 2013), we can give a definition, for each vertex $v$ linear transformations $P_v$ and each edge $e$ in $Q$, linear transformations $S_e$ and $S_e^\ast$ define as follows:

For all paths $x \in X$

1. $P_v(x) = \{ x, \text{if } x \in X_v \}
2. \begin{cases} \sigma_e(x), \text{if } x \in X_{r(e)} \end{cases}$
3. $0, \text{otherwise}$
4. $0, \text{otherwise}$
5. $0, \text{otherwise}$

It is easy to check that the endomorphisms $\{ p_v, s_e, S_e^\ast : v \in Q^0, e \in Q^1 \}$ fulfill relations (1)–(4) of the Leavitt path algebra $L$ (see Hazrat, 2013). This induces an algebra homomorphism $\varphi$ from $L$ to mapping $v$ to $P_v$, $e$ to $S_e$ and $e^\ast$ to $S_e^\ast$. Then we can construct $M(X)$ as a left module over $L$ through the homomorphism $\varphi$. We denote this $L$-module operation on by $\cdot$.

**Lemma 3.5.** If $u$ a sink or an infinite emitter, then module $A_u$ is a graded simple module which is simple.

Next, we will characterize graded simple modules which is simple for a Leavitt path algebras by using a quiver with a sink or infinite emitter.

By using Definition 3.1 and Lemma 3.5, we obtain the following theorem.

**Theorem 3.6.** If $v$ is a sink, then a graded simple module which is simple $A_v$ is not a completely prime module.

**Proof.** We can see Figure 1. Let $Q$ be the quiver in Figure 1.
The module $A_{v_3}$ which is simple is the module over $L$, with as basis the set $B = \{p : p \text{ paths in } Q \text{ with } r(p) = v_3\}$ and $v_3$ is a sink, $B$ actually only contains $\{v_3, e_1e_2, e_2, e_3, e_4e_3\}$. Let $r = -3v_1 + 3e_1 - 2e_2$, $r \in L$ and $m = v_3 + e_2 + e_3$, $m \in A_{v_3}$.

Then $rm = 0$. However, if we take $m_1 = e_2, m_1 \in A_{v_3}$ then $rm_1 \neq 0$. So $r \notin Ann A_{v_3}$.

Therefore $\exists m \neq 0$ and $r \notin A_{v_3}$, so $rm = 0$.

Thus, $A_{v_3}$ is not a completely prime module.

So the graded simple modules which is simple $A_{v_3}$ is not a completely module.

Furthermore, we can check in the case for $Q$ with subgraph in Figure 2:

![Figure 2. Subquiver Q with sink](image)

By using Definition 3.1 and Lemma 3.5, we also obtain the following theorem.

**Theorem 3.7.** If $v$ is an infinite emitter in $Q$, then graded simple modules which is simple $A_u$ is not a completely prime module.

**Proof.** Let see the quiver in Figure 3.

![Figure 3. Quiver Q with infinite emitter](image)

The simple module $A_{v_2}$ is the module over $L$, let the basis $B = \{e_1e_2, v_3\}$. Let $r = v_1 + 2e_1, r \in L$ and $m = v_2 - 2e_1 + e_2, m \in A_{v_2}$, thus $rm = 0$. However, if we take $m_1 = e_1, m_1 \in A_{v_2}$ then $rm_1 \neq 0$. So $r \notin Ann A_{v_2}$.

Therefore $\exists m \neq 0$ and $r \notin Ann A_{v_2}$ such that $rm = 0$.

Thus, $A_{v_2}$ is not a completely module.

**CONCLUSION**

This paper studied graded simple modules which are simple over Leavitt path algebras. We prove that if $u$ is a sink or infinite emitter, then a graded simple module which is simple $A_u$ is not a completely prime module.

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