The Characterization of Infinite Divisibility of Compound Negative Binomial Distribution as the Sum of Exponential Distribution

Anis Nur Afifah[1], Maiyastri[1], Dodi Devianto[1*]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Andalas University, Padang 25163, West Sumatra Province, INDONESIA*
Email: ddevianto@sci.unand.ac.id

Abstract
The sum of random variables that are identical and independent from an exponential distribution creates the compound distribution. It is called compound negative binomial distribution as the sum of exponential distribution when the number of random variables added follows the negative binomial distribution. This compound distribution's characteristic function is established by using mathematical analysis methods, included its uniform continuity property. The characteristic function's parametric curves never disappear from the complex plane, which means it is a positively defined function. Another characteristic function's property shows that this compound distribution is one of infinitely divisible distribution.

INTRODUCTION

The sum of random variables that are identical and independent where the number of random variables added follows a negative binomial distribution forms as the compound negative binomial distribution. These identical and independent random variables added follow an exponential distribution, it is called compound negative binomial distribution with the sum of exponential distribution. For an instant, suppose the random variable N has negative binomial distribution and $X_1, X_2,\ldots, X_N$ are identical and independent random variables from an exponential distribution. We denote the sum of these random variables by

$$S_N = X_1 + X_2 + ... + X_N \qquad (1)$$

as the compound distribution.

(Furman, 2007), who delivered the convolution of some independent random variables as the sum of negative binomial distribution and its properties, introduced the previous research on compound distribution.

The negative binomial-exponential model, as introduced by (Panjer & Willmot, 1981) is one of compound negative binomial distribution where its convolution is evaluated as finite sums. Furthermore, (Z. Wang, 2011) applied one mixed negative binomial distribution in the insurance. We have found another application of compound negative binomial distribution in many fields, such as it has done by (Rono et al., 2020) in modeling natural disaster in Kenya. (Girondot, 2017) used the convolution of negative binomial distribution to optimize the turtle research sampling design. (Omair et al., 2018) carried another application out by utilizing this distribution to develop a new bivariate model that is suitable for accident data.

This compound distribution has better theoretical development from statistical mathematics views, as explained by previous authors and its applications. (Chen & Guisong, 2017) derived the exact distribution using the generating function and the convolution, while (X. Wang et al., 2019) studied the characters of

compound negative binomial distribution got from the finite Markov chain approach. Furthermore, (Ong et al., 2019) described the various important properties like unmodelled, log-concavity, and asymptotic behavior from the compound negative binomial distribution and their use in practical applications.

The application of compound distribution as the convolution where the random variables added has an exponential distribution is widely used in insurance. We often assume the aggregate severity loss as an exponential distribution. Theoretically, (Devianto et al., 2015) proposed new distribution from the convolution of random variables that has an exponential distribution with stabilizer constant. They also suggested some essential characters from hypoexponential distribution with stabilizer constant. Moreover, (Devianto, 2016) explained the characteristic function's uniform continuity properties from the previously proposed distribution. (Devianto et al., 2019) discussed another study of the uniform continuity of other compound distribution's characteristic functions about Poisson compound distribution where the random variables added has Cauchy variational distribution.

It confirms that characteristic function has essential roles in the convolution and the compound distribution. The characterization of compound distribution with convolution from several special distributions is an important property to establish with its refinement in application. The leading theory in determining the properties of the characteristic function was proposed by (Lukacs, 1992) and developed by (Artikis, 1983) to construct an infinitely divisible distribution. This paper presents the characterization of infinite divisibility from a compound negative binomial distribution where the summed random variables have an exponential distribution.

METHOD

The research method in this paper is a literature study on the infinite divisibility property and compound negative binomial distribution's characterization gained from the sum of exponential distribution. First, we defined the characteristic function and other properties of the two constituent distributions, namely the negative binomial distribution and the exponential distribution. It is determined the sum of exponential distribution formed the compound negative binomial distribution then its properties are derived by using mathematical analysis based on its characteristic function. Parametric curves were used as a tool to show the properties of the characteristic functions. We examined it using Bochner's theorem to show that a function satisfies the condition and sufficient requirements as a characteristic function. The definition of the infinite divisibility of a distribution based on its characteristic function is used in the deductive method to prove that it is an infinitely divisible distribution.

RESULTS AND DISCUSSION

The density probability function of negative binomial distribution with parameter (r,p) for random variable N is defined as

$$f_N(n\,;r,p) = \binom{r+n-1}{n} p^r \ (1-p)^n \quad (2)$$

where $n = 0, 1, 2, ... \ r > 0$, and $p \in (0,1)$. The value of n represents the number of failures before a certain number of successes in the experiment. This specific number of successes denoted as r. We expressed the probability of success in an individual experiment as p, while the probability of failure is stated as $q = 1 - p$. Its expected value, variance, and moment generating function are written as follows, respectively:

$$E_N(N) = \frac{r(1-p)}{p}, \quad (3)$$

$$Var_N(N) = \frac{r(1-p)}{p^2}, \quad (4)$$

$$M_N(t) = \left(\frac{p}{1-(1-p)\exp(t)} \right)^r. \quad (5)$$

The other negative binomial distribution's properties can be determined by characteristic function. (Lukacs, 1992) and (Sato, 2013) have noted the characteristic function's definition using the Fourier-Stieltjes transform. We wrote the characteristic function of random variable X as

$$\phi_X(t) = E[\exp(itX)] \quad (6)$$

where $t \in (-\infty, \infty)$. The form $\exp(itX)$ can be written as $\cos tX + i \sin tX$ where $i = \sqrt{-1}$. So that the characteristic function for random variable N as negative binomial distribution is

$$\phi_N(t) = \left(\frac{p}{1-(1-p)\exp[it]} \right)^r. \qquad (7)$$

The shape of parametric curves shows the characterization of a characteristic function. The parametric curves are got by plotting the characteristic function at the Cartesian coordinates system. The real part of the characteristic function is represented as the *x*-axis, while we illustrate its imaginary part as the *y*-axis. For the negative binomial distribution, we plot it in Figure 1. Its characteristic function is uniformly continuous and for *p* tends to one, the characteristic function tends to one.

We defined the exponential distribution from a random variable *X* as a continuous probability distribution function with parameter λ as $f(x) = \lambda \exp(-\lambda x)$, where $\lambda > 0$ and $x > 0$. The expected value, variance, moment generating function, and its characteristic function are, respectively

$$E(X) = \frac{1}{\lambda}, \qquad (8)$$

$$Var(X) = \frac{1}{\lambda^2}, \qquad (9)$$

$$M_X(t) = \frac{\lambda}{\lambda - t}, \qquad (10)$$

$$\phi_X(t) = \frac{\lambda}{\lambda - it}. \qquad (11)$$

The parametric curves of the exponential distribution's characteristic function with various parameters λ are shown in Figure 2. It is uniformly continuous and is a circle with a radius of 0.5 for all of various λ.

Suppose that random variable *N* has negative binomial distribution with parameter (*r,p*). Let $X_1, X_2, ..., X_N$ be identical and independent random variables from an exponential distribution with parameter λ. These summed random variables construct a new random variable that has negative binomial compound distribution with the sum of exponential distribution $S_N = X_1 + X_2 + ... + X_N$ contains the parameter (*r,p,λ*). This distribution's characterization is a critical theory to determine since it's widely used in insurance modeling for the severity lost model. The following propositions give some refinement properties of compound negative binomial distribution related to uniform continuity and positively defined function.
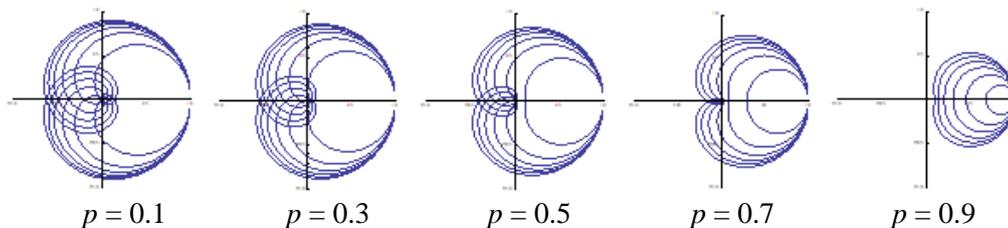

| $p = 0.1$ | $p = 0.3$ | $p = 0.5$ | $p = 0.7$ | $p = 0.9$ |

Figure 1. Parametric curves of negative binomial distribution's characteristic function with various parameters *p* and $r = 2, 4, 6, 8, 10, 12$ for each graph.
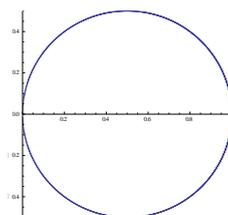

Figure 2. Parametric curves of exponential distribution's characteristic function for various parameters λ = 0.1, 0.5, 1, 2, 4, 8.

Proposition 1. *Suppose that S is a generated random variable from the sum of exponential distribution as the compound negative binomial distribution and has parameters* (r, p, λ)*. Its expectation and variance are defined as.*

$$E(S) = \frac{r(1-p)}{\lambda p}, \qquad (12)$$

$$Var(S) = \frac{r(1-p)(1+p)}{(\lambda p)^2}. \qquad (13)$$

Proof. The definition of this generated random variable and the linearity property of expectation are used to get the expectation of this compound distribution, it is.

$$E(S) = E\left(\sum_{i=1}^{N} X_i\right) = E\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right)$$

$$= E(N\,E(X)) = E(N)E(X) = \frac{r(1-p)}{\lambda p}. \qquad (14)$$

From the expectation of negative binomial distribution in (3) and expectation of exponential distribution in (8), we have a proof for equation (12). While by using the definition of variance and linearity of expectation, then it is obtained

$$Var(S) = E\left(Var\left(\sum_{i=1}^{N} X_i \middle| N\right)\right) + Var\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right)$$

$$= E(N)Var(X) + Var(N)(E(X))^2 = \frac{r(1-p)(1+p)}{(\lambda p)^2}. \qquad (15)$$

By taking expectation from negative binomial distribution and exponential distribution in (3) and (8) also its variance in (4) and (9), we have a proof for equation (13).■

Proposition 2. *Suppose that S is a generated random variable from the sum of exponential distribution as the compound negative binomial distribution and has parameters* (r, p, λ)*. Its*

moment generating function and characteristic function are defined as, respectively:

$$M_S(t) = \left(\frac{p\lambda - pt}{p\lambda - t}\right)^r, \qquad (16)$$

$$\phi_S(t) = \left(\frac{p}{1 - (1-p)\dfrac{\lambda}{\lambda - it}}\right)^r. \qquad (17)$$

Proof. By applying the definition of compound distribution's moment generating function and the linearity property of its expectation, we get for independent and identically random variable S as follows.

$$M_S(t) = E(M_X(t)^N) = M_N(\ln(M_X(t)))$$

$$= \left(\frac{p\lambda - pt}{p\lambda - t}\right)^r. \qquad (18)$$

It is used moment generating function from negative binomial and exponential distribution in (5) and (10) then we have a proof for equation (16). While by using the expectation's linearity property, then we get

$$\phi_S(t) = E(\phi_X(t)^N) = M_N(\ln(\phi_X(t)))$$

$$= \left(\frac{p}{1 - (1-p)\dfrac{\lambda}{\lambda - it}}\right)^r. \qquad (19)$$

It is used the negative binomial distribution's moment generating function in (5) and exponential distribution's characteristic function in (11), so equation (17) is proven. ■

The characteristic function's characterization from this compound negative binomial distribution for various parameters is shown by parametric curves in Figure 3, Figure 4, Figure 5, and Figure 6.

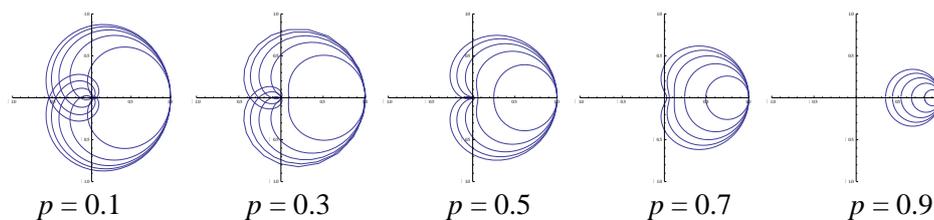$p = 0.1$ $p = 0.3$ $p = 0.5$ $p = 0.7$ $p = 0.9$

Figure 3. Characteristic function's parametric curves from compound distribution as negative binomial with the sum of an exponential distribution for various parameters *p* where $r = 2, 4, 6, 8, 10$ and $\lambda = 0.5$ in each graph.

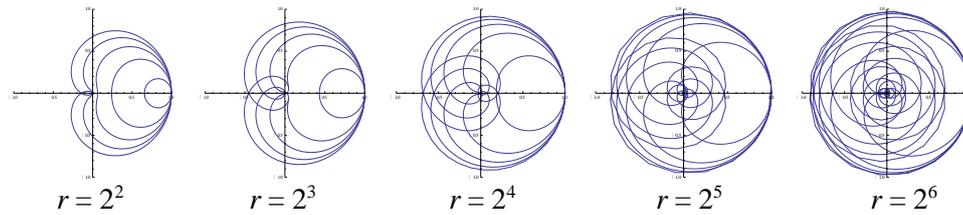$$r = 2^2 \qquad r = 2^3 \qquad r = 2^4 \qquad r = 2^5 \qquad r = 2^6$$

Figure 4. Characteristic function's parametric curves from compound distribution as negative binomial with the sum of an exponential distribution for various parameters r where $p = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\lambda = 0.5$ in each graph.
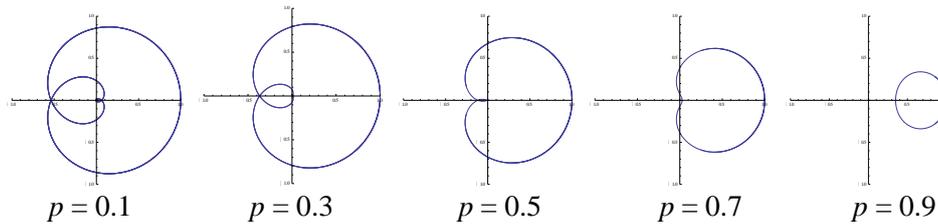

$$p = 0.1 \qquad p = 0.3 \qquad p = 0.5 \qquad p = 0.7 \qquad p = 0.9$$

Figure 5. Characteristic function's parametric curves from compound distribution as negative binomial with the sum of an exponential distribution for various parameters p where $\lambda = 2, 4, 8, 16, 32$ and $r = 10$ in each graph.

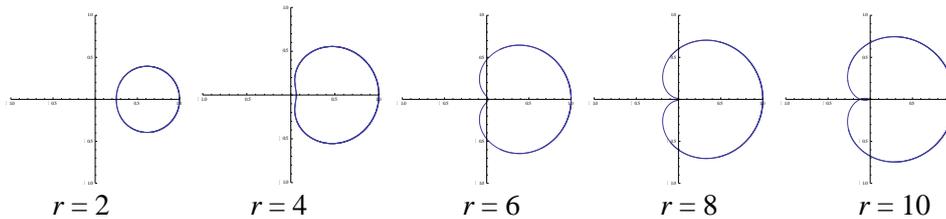$$r = 2 \qquad r = 4 \qquad r = 6 \qquad r = 8 \qquad r = 10$$

Figure 6. Characteristic function's parametric curves from compound distribution as negative binomial with the sum of an exponential distribution for various parameters r where $\lambda = 2, 4, 8, 16, 32$ and $p = 0.5$ in each graph.

The characteristic function from a random variable that follows the compound distribution described in (1), as shown in the parametric curve above, is uniformly continuous. Its characteristic function tends to one for various parameters λ and r when parameter p tends to one, such as in Figure 3 and Figure 5. The greater parameter r then more complex graphs of parametric curves such as in Figure 4, while the shape of the parametric curve does not change for various parameter λ as in Figure 6. This parametric curve's shape never disappears from the complex-plane and always depends on the parameter (r, p, λ). It is more complicated than the representation of characteristic function from the negative binomial distribution or exponential distribution.

The next propositions will tell the major result of characteristic function properties from a compound negative binomial distribution with some of exponential distribution. A characteristic function from a distribution is a function with complex values defined in Bochner's theorem, as explained by (Kendall & Stuart, 1961). The definition says that a function $\varphi(t)$ that maps the real number t to the complex value is called characteristic function if in any case, $\varphi(t)$ is equal to 1 when $t = 0$ and $\varphi(t)$ is a non-negative definite function.

The definition of non-negative definite function is when it satisfies two terms that are a continuous function and the sum $\sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \varphi_S(t_j - t_l) \ge 0$ is real and non-negative

for any number n which is positive integer, any real number $t_1, t_2, ..., t_n$ and any complex number $c_1, c_2, ..., c_n,$.

Proposition 3. *The characteristic $\phi_S(t)$ contains the parameter (r, p, λ) as written in (17) from a random variable S as compound negative binomial distribution with the sum of exponential distribution satisfies*

(i) $\phi_S(0) = 1$;

(ii) $\phi_S(t)$ *is uniformly continuous;*

(iii) $\phi_S(t)$ *is a positively defined function with quadratic form*

$$\sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \phi_X(t_j - t_l) \ge 0 \qquad (20)$$

for real number $t_1, t_2, ..., t_n$ and any complex number $c_1, c_2, ..., c_n$.

Proof. *(i)* It is easy to obtain $\phi_S(0) = 1$.

(ii) The uniform continuity of $\phi_S(t)$ is gained by using the definition of continuity. The definition says that for any $\varepsilon > 0$, always there is a $\delta > 0$ so for $|t_1 - t_2| < \delta$ then $|\phi_S(t_1) - \phi_S(t_2)| < \varepsilon$. The value of δ only depends on ε. Therefore, we get :

$$|\phi_S(t_1) - \phi_S(t_2)|$$

$$= p^r \left| \left(\frac{1}{1 - (1-p)\phi_X(t_1)} \right)^r - \left(\frac{1}{1 - (1-p)\phi_X(t_2)} \right)^r \right|.$$
$$(21)$$

Then suppose $h = t_1 - t_2$, so for $h \to 0$, the limit of equation (21) is approaching zero like written below:

$$\left| \left(\frac{1}{1 - (1-p)\phi_X(h + t_2)} \right)^r - \left(\frac{1}{1 - (1-p)\phi_X(t_2)} \right)^r \right| \to 0.$$
$$(22)$$

This equation holds for $\delta < \varepsilon$ and $|t_1 - t_2| < \delta$ where $|\phi_S(h + t_2) - \phi_S(t_2)| < \varepsilon$. Then $\phi_S(t)$ is uniformly continuous.

(iii) We will prove that the characteristic function $\phi_S(t)$ is a positively defined function that has quadratic form written in equation (20). Using the characteristic function in equation (17) and geometric series where the ratio is

$(1 - p)E(\exp(i(t_j - t_l)X))$ for $p \in (0,1)$ and $E(\exp(i(t_j - t_l)X)) \le 1$, we have

$$\sum_{1 \le j, l \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \phi_S(t_j - t_l)$$

$$= \sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \left(\frac{p}{1 - (1-p)\dfrac{\lambda}{\lambda - i(t_j - t_l)}} \right)^r$$

$$= \sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \left(\frac{p}{1 - (1-p)E(\exp(i(t_j - t_l)X))} \right)^r$$

$$= p^r \sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \left(\sum_{m=0}^{\infty} (1-p)E(\exp(i(t_j - t_l)X)) \right)^{mr}.$$
$$(23)$$

Using the conjugate properties in the complex number, so we gain the following equation.

$$\sum_{1 \le j \le n} \sum_{1 \le l \le n} c_j \overline{c_l} \phi_S(t_j - t_l)$$

$$= p^r \underline{\sum_{1 \le j \le n} c_j \left(\sum_{m=0}^{\infty} (1-p)E(\exp(it_j X)) \right)^{mr}}$$
$$\overline{\sum_{1 \le l \le n} c_l \left(\sum_{m=0}^{\infty} (1-p)E(\exp(it_l X)) \right)^{mr}}$$

$$= p^r \left| \sum_{1 \le j \le n} c_j \left(\sum_{m=0}^{\infty} (1-p)E(\exp(it_j X)) \right)^{mr} \right|^2$$

$$= p^r \left| \sum_{1 \le j \le n} c_j \left(\sum_{m=0}^{\infty} (1-p) \frac{\lambda}{\lambda - i \; t_j} \right)^{mr} \right|^2 \ge 0.$$
$$(24)$$

So $\phi_S(t)$ is proved as a positively defined function with the quadratic form as written in (20) that has non-negative values.■

We can conclude that $\phi_S(t)$ is a characteristic function from random variable S which is generated from the sum of exponential distribution and follows the compound negative binomial distribution where the parameters are (r, p, λ). When $t = 0$, its value is equal to 1. It is a positively defined function that never disappears from the complex plane, such as proven in Proposition 3 and shown in the parametric curve at Figures 3-6.

Proposition 4. *The function*

$$\phi_{S_n}(t) = \left(\frac{p}{1 - (1-p)\dfrac{\lambda}{\lambda - it}} \right)^{\frac{r}{n}} \qquad (25)$$

is a characteristic function.

Proof. Based on Proposition 3, then it satisfies the characteristic function's criteria of Bochner's Theorem. It is easy to say that the function $\phi_{S_n}(t)$ is a characteristic function from random variable S_n, which is gained from the sum of exponential distribution, and follows the compound negative binomial distribution, which has parameters $(r/n, p, \lambda)$.

Proposition 5. *The sum of exponential distribution which construct the new compound negative binomial distribution is infinitely divisible.*

Proof. We use it the definition of infinite divisibility in (Artikis, 1983) to prove this part. It will show that for every n, which is positive integer number, there exist $\phi_{S_n}(t)$, a characteristic function that satisfies $\phi_S(t) = (\phi_{S_N}(t))^n$. Based on Proposition 4, we have $\phi_{S_n}(t)$ in equation (25) so that

$$(\phi_{S_n}(t))^n = \left(\frac{p}{1-(1-p)\frac{\lambda}{\lambda - it}} \right)^r = \phi_S(t)$$

(26)

We obtain $\phi_S(t)$ as a characteristic function from compound negative binomial distribution with the sum of exponential distribution which has parameters (r, p, λ). Because of its fulfillment the condition of $\phi_S(t) = \left(\phi_{S_n}(t) \right)^n$ for each t, so that the characteristic function is infinitely divisible.∎

CONCLUSION

This paper introduces the new compound negative binomial distribution, which is generated from the sum of exponential distribution. Its characterization is gained by applying and combining some properties of negative binomial distribution and exponential distribution. It is using mathematical analysis methods to explore its characteristic function and show uniform continuity property. The plot of its characteristic function as a parametric curve never disappears from the complex plane. It is also a positively defined function. This compound's characteristic function satisfies the definition of an infinitely divisible distribution.

REFERENCES

Artikis, T. (1983). Constructing infinitely divisible characteristic functions. *Archivum Mathematicum*, *19*(2), 57–61.

Chen, X., & Guisong, C. (2017). Exact distribution of the convolution of negative binomial random variables. *Communications in Statistics - Theory and Methods*, *46*(6), 2851–2856. https://doi.org/10.1080/03610926.2015.1053931

Devianto, D. (2016). The uniform continuity of characteristic function from convoluted exponential distribution with stabilizer constant. *AIP Conference Proceedings*, *1707*. https://doi.org/10.1063/1.4940863

Devianto, D., Oktasari, L., & Anas, M. (2015). Convolution of Generated Random Variable from Exponential Distribution with Stabilizer Constant. *Applied Mathematical Sciences*, *9*(96), 4781–4789. https://doi.org/10.12988/ams.2015.55396

Devianto, D., Sarah, H., Yozza, F., & Yanuar, M. (2019). The Infinitely Divisible Characteristic Function of Compound Poisson Distribution as the Sum of Variational Cauchy Distribution. *Journal of Physics: Conference Series*, *1397*, 12065. https://doi.org/10.1088/1742-6596/1397/1/012065

Furman, E. (2007). On the convolution of the negative binomial random variables. *Statistics and Probability Letter*, *77*, 169–172.

Girondot, M. (2017). Optimizing sampling design to infer the number of marine turtles nesting on low and high density sea turtle rookeries using convolution of negative binomial distribution. *Ecological Indicators*, *81*, 83–89. https://doi.org/10.1016/j.ecolind.2017.05.063

Kendall, M. G., & Stuart, A. (1961). *The Advanced Theory of Statistics. Third Edition*. Hafner Publishing Company.

Lukacs, E. (1992). *Characteristic Function*. Hafner Publishing Company.

Omair, M., Almuhayfith, F., & Alzaid, A. (2018). A bivariate model based on compound negative binomial distribution. *Revista Colombiana de Estadística*, *41*(1),

87–108.
https://doi.org/10.15446/rce.v41n1.57803

Ong, S. H., Toh, K., & Low, Y. C. (2019). *The non-central negative binomial distribution: Further properties and applications*. Communications in Statistics-Theory and Methods.

Panjer, H. H., & Willmot, G. E. (1981). Finite sum evaluation of the negative binomial-exponential model. *Astin Bulletin*, 133–137.

Rono, A., Ogutu, C., & Weke, P. (2020). On Compound Distributions for Natural Disaster Modelling in Kenya. *International Journal of Mathematics and Mathematical Sciences*, *Volume 202*. https://www.hindawi.com/journals/ijmms /2020/9398309/

Sato, K. (2013). *Levy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advance Mathematics.

Wang, X., Zhao, X., & Sun, J. (2019). A compound negative binomial distribution with mutative termination conditions based on a change point. *Journal of Computational and Applied Mathematics*, *351*, 237–249.

Wang, Z. (2011). One mixed negative binomial distribution with application. *Journal of Statistical Planning and Inference*, *141*, 1153–1160. https://www.sciencedirect.com/science/ar ticle/pii/S0378375810004313